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On the nonlinear evolution of three-dimensional disturbances in plane Poiseuille flow

By A. DAVEY,
School of Mathematics, University of Newcastle upon Tyne
L. M. HOCKING AND K. STEWARTSON
Department of Mathematics, University College, London

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The equations governing the nonlinear development of a centred three-dimensional disturbance to plane parallel flow at slightly supercritical Reynolds numbers are obtained. In contrast to the corresponding equation for two-dimensional disturbances, two slowly varying functions are needed to describe the development: the amplitude function and a function related to the secular pressure gradient produced by the disturbance. These two functions satisfy a pair of coupled partial differential equations. The equations derived in Hocking, Stewartson & Stuart (1972) are shown to be incorrect. Some of the properties of the governing equations are discussed briefly.

1. Introduction

A theory of the evolution of an infinitesimal centred disturbance in plane Poiseuille flow (fully developed steady flow under a constant pressure gradient between fixed parallel planes) at a Reynolds number slightly greater than the critical value for stability was recently developed by Stewartson & Stuart (1971). In this paper, subsequently referred to as SS, the disturbances were two-dimensional and it was established that, when nonlinear effects first become significant, the amplitude of the disturbance satisfies a nonlinear Schrödinger equation with complex coefficients. The properties of solutions of this equation were examined in a series of papers (Hocking *et al.* 1972 (HSS); Hocking & Stewartson 1972 (HS1), 1971(HS2)) and the theory was extended to include three-dimensional disturbances.

Unfortunately, the three-dimensional theory contains an error and an additional term must be added to the generalized Schrödinger equation. This term contains a new function, which is related to the amplitude function by Poisson's equation. Thus the evolution of the disturbance becomes much more difficult to follow, even numerically, although, since the numerical coefficient multiplying the new term is relatively small, it is likely that any modifications of the original conclusions will be only quantitative. This is certainly the case for disturbances which are in the form of oblique plane waves, for which the original equation may be recovered, except for a change in the values of the complex coefficients of its terms.

The linear terms in the amplitude equation were correctly obtained in the previous work, by noting that they must be of such a form that small amplitude solutions satisfy the dispersion relation for linear waves in the basic flow. The nonlinear terms in the two-dimensional theory are the same as those which are present in the corresponding equation for the development of a modal disturbance, and it was thought that the same would be true in the three-dimensional theory. However, there are *two* slowly varying functions needed to characterize the disturbance: the amplitude and a secular pressure term. This pressure was not explicitly determined in the two-dimensional theory, since the analysis was performed in terms of the stream function and the pressure formally eliminated. The contribution of the secular pressure to the nonlinear part of the amplitude equation was combined with that arising from other sources. In the three-dimensional theory, however, the dependence of the secular pressure on two slowly varying spatial co-ordinates prevents its elimination, and two governing equations are required. The importance of this pressure term in nonlinear stability theory was foreshadowed by Stuart (1958) and its significance has been made explicit in recent studies by Eagles (1971) and by Stuart & DiPrima (1974).

In this paper we derive the correct form of the equations governing the development of centred three-dimensional disturbances and discuss briefly some of their properties.

2. Three-dimensional nonlinear disturbances

We broadly follow the notation of SS. Suppose that the two planes are a distance $2h$ apart and let O be any convenient point midway between them. Let $Oxyz$ be a system of Cartesian co-ordinates in which Oz is perpendicular to the planes and Ox is in the direction of the undisturbed stream; further, let hx, hy and hz measure distances parallel to the axes, and let $U_0\mathbf{v}$ be the fluid velocity, where U_0 is the maximum velocity of the undisturbed stream and where $\mathbf{v}$ has components (u, v, w). The governing equations of motion for an incompressible fluid are then

$$\nabla.\mathbf{v} = 0, \quad \partial\mathbf{v}/\partial t + (\mathbf{v}.\nabla)\,\mathbf{v} = -\nabla P + R^{-1}\nabla^2\mathbf{v}, \tag{2.1}$$

where $R = U_0h/\nu$ is the Reynolds number, ν is the kinematic viscosity and P is the non-dimensional pressure. The corresponding boundary conditions are that

$$u = v = w = 0 \quad \text{at} \quad z = \pm 1. \tag{2.2}$$

In the undisturbed motion

$$u = 1 - z^2, \quad v = w = 0, \quad dP/dx = -2/R, \tag{2.3}$$

which is the fully developed flow produced by a uniform pressure gradient, and we shall assume that, even when the flow is disturbed, (2.3) holds as $x^2 + y^2 \to \infty$. The linear theory of stability proceeds by assuming that u, v, w and P are each slightly perturbed from their steady-state values by expressions of the form $\exp\{i\alpha(x - ct) + i\beta y\}$ multiplied by a function of z, where α and β are real. Non-trivial solutions of the linearized forms of (2.1) which then result are only possible if α, β, R and c are related. Although there are an infinite number of values of c for any fixed α, β and R, we concentrate attention on that value of $c = c_r + ic_i$ which has the maximum imaginary part and is therefore the one most likely to lead to instability. For this c, there is a critical value R_c of R such that, if $R < R_c$, $c_i < 0$ for all α and β and if $R = R_c$, $c_i = 0$ at the single point $(\alpha_c, 0)$ in α, β space. If $R > R_c$, there is a domain of α, β space where $c_i \geqslant 0$. If $c = c_{cr}$ at the critical point $R = R_c$, $\alpha = \alpha_c$, $\beta = 0$, we can expand the complex growth rate of the linearized disturbance in the vicinity of the critical point in the form

$$-i\alpha c = -i\alpha_c c_{cr} + ia_{1r}(\alpha - \alpha_c) - a_2(\alpha - \alpha_c)^2 - b_2\beta^2 + (R - R_c)\,d_1 + \dots, \tag{2.4}$$

where

$$\left.\begin{aligned} R_c = 5772{\cdot}22, \quad \alpha_c = 1{\cdot}02055, \quad c_{cr} = 0{\cdot}264, \quad a_{1r} = -0{\cdot}383,\\ a_2 = 0{\cdot}187 + 0{\cdot}0275i, \quad d_1 = (0{\cdot}168 + 0{\cdot}811i)\,10^{-5}. \end{aligned}\right\} \tag{2.5}$$

The value of b_2 can be found by an application of Squire's (1933) theorem, which yields the expression

$$b_2 = \frac{d_1 R_c}{2\alpha_c^2} - \frac{i(a_{1r} + c_{cr})}{2\alpha_c} = 0{\cdot}00466 + 0{\cdot}0808i. \tag{2.6}$$

The value of R_c agrees, to the accuracy quoted, with the value obtained by Orszag (1971).

These numbers have all been recalculated and are in very good agreement with the values calculated by Dr R. R. Cousins and used in the previous set of papers, with the exception of a_2, for which Dr Cousins obtained the value $0{\cdot}183 + 0{\cdot}070i$. The value quoted in (2.5) has been calculated by two methods, one making use of the formula for a_2 given in SS and the other using (2.4) and calculations of the values of c near the critical point. Also, a recalculation by Dr Cousins (private communication) using the second method gave values of a_{1r}, a_2 and d_1 in agreement with ours.

The nonlinear theory of wave systems, as developed in SS, begins by supposing that an infinitesimal disturbance, centred at O but otherwise arbitrary, is made to the solution (2.3) at a Reynolds number R slightly greater than R_c. After a long time has elapsed, the disturbance will have evolved so that its Fourier decomposition only contains waves in which $\alpha \approx \alpha_c$ and $\beta \approx 0$, and of these only some will be growing in amplitude. Thus the linear theory filters out all but a small group of waves, which travels downstream with the group velocity $-a_{1r}$ and which spreads out from its centre to a distance $O(t^{\frac{1}{2}})$. This group is affected by nonlinearity and its subsequent evolution can be found using the method of multiple scales.

We define, as in HSS,

$$\left.\begin{aligned} \epsilon = (R - R_c)\,d_{1r}, \quad E = \exp\{i\alpha_c(x - c_{cr}t)\},\\ \tau = \epsilon t, \quad \xi = \epsilon^{\frac{1}{2}}(x + a_{1r}t), \quad \eta = \epsilon^{\frac{1}{2}}y, \end{aligned}\right\} \tag{2.7}$$

and write

$$u = u_0(\xi, \eta, \tau, z; \epsilon) + Eu_1(\xi, \eta, \tau, z; \epsilon) + E^{-1}\tilde{u}_1 + E^2u_2 + E^{-2}\tilde{u}_2 + \dots, \tag{2.8}$$

where a tilde denotes the complex conjugate, with similar expressions for v, w and P. We also write

$$\begin{aligned} u_0 &= 1-z^2 + \epsilon u_{02}(\xi,\eta,\tau,z) + \epsilon^{\frac{3}{2}} u_{03} + \ldots, \\ u_1 &= \epsilon^{\frac{1}{2}} u_{11}(\xi,\eta,\tau,z) + \epsilon u_{12}(\xi,\eta,\tau,z) + \epsilon^{\frac{3}{2}} u_{13} + \ldots, \\ u_2 &= \epsilon u_{22}(\xi,\eta,\tau,z) + \epsilon^{\frac{3}{2}} u_{23} + \ldots, \\ u_3 &= \epsilon^{\frac{3}{2}} u_{33} + \ldots, \end{aligned} \tag{2.9}$$

and so on, with similar expressions for v, w and P, except that v_0 and w_0 do not contain the term $1-z^2$ and

$$P_0 = -2R^{-1}x + \text{constant} + \epsilon^{\frac{1}{2}} P_{01} + \epsilon P_{02} + \ldots. \tag{2.10}$$

The above expressions are now substituted into the governing equations (2.1) and coefficients of $\epsilon^{\frac{1}{2}n}E^m$ $(n, m = 0, 1, 2 \ldots)$ equated to zero. From the coefficient of $\epsilon^{\frac{1}{2}}E$ we obtain

$$u_{11} = A(\xi,\eta,\tau)\, D\psi_1(z), \quad v_{11} = 0, \quad w_{11} = -i\alpha_c A\psi_1(z), \tag{2.11}$$

where $D = \partial/\partial z$ and ψ_1 is the eigenfunction of the Orr–Sommerfeld equation

$$\mathscr{L}\psi_1 \equiv [(i/\alpha_c R_c)(D^2-\alpha_c^2)^2 + (1-z^2-c_{cr})(D^2-\alpha_c^2) + 2]\,\psi_1 = 0, \tag{2.12}$$

normalized so that $\psi_1(0) = 1$. The determination of the amplitude function A is the main purpose of the present nonlinear stability theory.

From the coefficient of ϵE we get

$$u_{12} = -i\frac{\partial A}{\partial \xi} D\psi_{10} + A_2 D\psi_1, \quad w_{12} = -\frac{\partial A}{\partial \xi}(\alpha_c \psi_{10} + \psi_1) - i\alpha_c A_2 \psi_1, \tag{2.13}$$

where $\psi_{10}(z)$ satisfies

$$\begin{aligned} \mathscr{L}\psi_{10} &= -\alpha_c^{-1}[(1-z^2+a_{1r}-4i\alpha_c R_c^{-1})(D^2-\alpha_c^2) - 2\alpha_c^2(1-z^2-c_{cr}) + 2]\,\psi_1 \\ &\equiv -\alpha_c^{-1}\mathscr{M}\psi_1 \end{aligned} \tag{2.14}$$

and A_2 is another function of ξ, η and τ. The existence of a solution of (2.14) satisfying the boundary conditions $\psi_{10}(\pm 1) = D\psi_{10}(\pm 1) = 0$ is assured by the correct choice of a_{1r}, as in SS (2.21). Because of the extension to three-dimensional disturbances, there is also a y component of velocity, given by

$$v_{12} = (\partial A/\partial \eta)\,\chi_1(z), \tag{2.15}$$

where
$$\{D^2-\alpha_c^2 - i\alpha_c R_c(1-z^2-c_{cr})\}(i\alpha_c\chi_1 - D\psi_1) + 2i\alpha_c R_c z\psi_1 = 0 \tag{2.16}$$

and $\chi_1(\pm 1) = 0$; it is not necessary to compute χ_1 explicitly in order to determine A.

From the coefficient of ϵE^2 we get

$$u_{22} = A^2 D\psi_2, \quad v_{22} = 0, \quad w_{22} = -2i\alpha_c A^2\psi_2, \tag{2.17}$$

where $\psi_2(z)$ is a function of z defined in SS (3.10) and SS (3.12) and computed by Reynolds & Potter (1967).

The coefficient of $\epsilon^{\frac{1}{2}}E^0$ gives

$$\partial P_{01}/\partial z = 0, \tag{2.18}$$

and from the coefficient of ϵE^0 we obtain

$$D^2u_{02}-R_c\partial P_{01}/\partial\xi = i\alpha_c R_c|A|^2D\{\tilde{\psi}_1 D\psi_1-\psi_1 D\tilde{\psi}_1\}, \tag{2.19}$$

$$D^2v_{02}-R_c\partial P_{01}/\partial\eta = 0 \tag{2.20}$$

and, from the equation of continuity,

$$Dw_{02} = 0. \tag{2.21}$$

Also, the coefficient of $\epsilon^{\frac{3}{2}}E^0$ in the equation of continuity is

$$\partial u_{02}/\partial\xi+\partial v_{02}/\partial\eta+Dw_{03} = 0, \tag{2.22}$$

and, on applying the boundary condition $w = 0$ at $z = \pm 1$, we obtain

$$w_{02} = 0, \quad \int_{-1}^{1}\left(\frac{\partial u_{02}}{\partial\xi}+\frac{\partial v_{02}}{\partial\eta}\right)dz = 0. \tag{2.23}$$

From these equations we find

$$u_{02} = -\tfrac{1}{2}(1-z^2)R_c\partial P_{01}/\partial\xi+|A|^2S(z), \tag{2.24}$$

$$v_{02} = -\tfrac{1}{2}(1-z^2)R_c\partial P_{01}/\partial\eta, \tag{2.25}$$

where

$$S(z) = i\alpha_c R_c\int_1^z(\tilde{\psi}_1 D\psi_1-\psi_1 D\tilde{\psi}_1)\,dz, \tag{2.26}$$

and

$$\frac{\partial^2 P_{01}}{\partial\xi^2}+\frac{\partial^2 P_{01}}{\partial\eta^2} = \frac{3}{R_c}\frac{\partial|A|^2}{\partial\xi}\int_0^1 S(z)\,dz. \tag{2.27}$$

The difference between the two- and three-dimensional theories can now be seen. When A and P_{01} are independent of η, the pressure gradient $\partial P_{01}/\partial\xi$ and thus the velocity component u_{02} are proportional to $|A|^2$, but such a simplification is not possible in the three-dimensional theory. The error in the previous work was a result of overlooking the spanwise pressure gradient $\partial P_{01}/\partial\eta$. To facilitate comparison between the two- and three-dimensional theories, we write

$$B(\xi,\eta,\tau) = |A|^2-\tfrac{1}{3}R_c\left(\int_0^1 S(z)\,dz\right)^{-1}\frac{\partial P_{01}}{\partial\xi}, \tag{2.28}$$

so that

$$u_{02} = |A|^2DF(z)+\left(\frac{3}{2}\int_0^1 S(z)\,dz\right)(1-z^2)B, \tag{2.29}$$

where

$$DF(z) = S(z)-\tfrac{3}{2}(1-z^2)\int_0^1 S(z)\,dz, \tag{2.30}$$

as defined in SS (3.14). The additional function B satisfies

$$\frac{\partial^2 B}{\partial\xi^2}+\frac{\partial^2 B}{\partial\eta^2} = \frac{\partial^2}{\partial\eta^2}|A|^2, \tag{2.31}$$

and is zero for unskewed two-dimensional disturbances. A numerical computation gave

$$\int_0^1 S(z)\,dz = -87{\cdot}2, \tag{2.32}$$

and the same value was also obtained from results kindly supplied by Prof. W. C. Reynolds.

We can now proceed to the coefficient of $\epsilon^{\frac{3}{2}}E$. The x and z components of the momentum equation and the equation of continuity can be reduced to a single equation for w_{13}, by eliminating the pressure and the function χ_1 by means of (2.16). The equation can be written as

$$\begin{aligned}\mathscr{L}w_{13} = {} & \frac{\partial A}{\partial \tau}(D^2-\alpha_c^2)\psi_1 + \frac{i\alpha_c A}{d_{1r}R_c}[(1-z^2-c_{cr})(D^2-\alpha_c^2)+2]\psi_1 \\ & + \frac{\partial A_2}{\partial \xi}\mathscr{M}\psi_1 - i\frac{\partial^2 A}{\partial \xi^2}\left(\mathscr{M}\psi_{10} + \frac{1}{\alpha_c}\mathscr{M}\psi_1\right) - \frac{\partial^2 A}{\partial \xi^2}[2R_c^{-1}(D^2-3\alpha_c^2) \\ & - i\alpha_c(3-3z^2+2a_{1r}-c_{cr})]\psi_1 - \frac{\partial^2 A}{\partial \eta^2}[2R_c^{-1}(D^2-\alpha_c^2) - i\alpha_c(1-z^2-c_{cr})]\psi_1 \\ & - i\alpha_c A|A|^2[D\psi_2(D^2-\alpha_c^2)\tilde{\psi}_1 + 2\psi_2(D^2-\alpha_c^2)D\tilde{\psi}_1 - 2D\tilde{\psi}_1(D^2-4\alpha_c^2)\psi_2 \\ & - \tilde{\psi}_1(D^2-4\alpha_c^2)D\psi_2 - DF(D^2-\alpha_c^2)\psi_1 + \psi_1 D^3F] \\ & + i\alpha_c AB\left(\frac{3}{2}\int_0^1 S(z)\,dz\right)[(1-z^2)(D^2-\alpha_c^2)+2]\psi_1. \end{aligned} \tag{2.33}$$

Since the operator $\mathscr{L}$ has an eigensolution satisfying the appropriate boundary conditions at $z = \pm 1$ (which are $w_{13}(\pm 1) = Dw_{13}(\pm 1) = 0$, from (2.2) and the equation of continuity), a solution of (2.33) is only possible if the right-hand side satisfies a certain integral condition. This can most simply be found by multiplying it by the adjoint function Φ of $\mathscr{L}$ and integrating from -1 to 1. Graphs of the real and imaginary parts of Φ are given by Reynolds & Potter (1967). Fortunately, the various integrals required have already been computed or can be expressed in terms of d_1 using SS (2.2). We find that

$$\frac{\partial A}{\partial \tau} - a_2\frac{\partial^2 A}{\partial \xi^2} - b_2\frac{\partial^2 A}{\partial \eta^2} = \frac{d_1}{d_{1r}}A + k|A|^2A + qAB, \tag{2.34}$$

where
$$k = 30{\cdot}8 - 173i, \tag{2.35}$$

using Reynolds & Potter's results, and

$$q = \tfrac{3}{2}(R_c d_1 - i\alpha_c c_{cr})\int_0^1 S(z)\,dz = -1{\cdot}27 + 29{\cdot}1i. \tag{2.36}$$

The specification of A and B may now be formally completed with the initial and boundary conditions. From HSS (2.22) we have, for disturbances which are centred at O at $t = 0$,

$$A \approx \frac{\Delta\epsilon^{\frac{1}{2}}}{\tau}\exp\left(-\frac{\xi^2}{4a_2\tau} - \frac{\eta^2}{4b_2\tau}\right) \quad \text{as} \quad \tau \to 0, \tag{2.37}$$

where Δ is a constant determined by the original amplitude of the disturbance. Since the disturbance is assumed to die away sufficiently far from its centre, we also have

$$|A|,\ |B| \to 0 \quad \text{as} \quad \xi^2 + \eta^2 \to \infty. \tag{2.38}$$

3. Discussion

The pair of differential equations (2.31) and (2.34) which must be solved in order to determine the evolution of a three-dimensional disturbance in the non-linear regime have been obtained for plane Poiseuille flow. It is likely that a similar pair of equations may be obtained for other marginally unstable shear flows. Indeed the evolution of a wave packet on the surface of water is governed by such equations, only the coefficients of the various terms being changed (Davey & Stewartson 1974), even though there is no basic motion of the water, let alone a shear flow. In both cases the nonlinear terms in the equations arise in part from the secular motion of the fluid, due to the disturbance and described by the velocity components u_{02} and v_{02}. Notwithstanding the large value of R, this motion is very slow and is in fact an example of Hele–Shaw flow.

Because of the additional term in (2.34), the three-dimensional studies reported in HS2 are strictly relevant only to flows for which $q = 0$, but they may be useful as guides to the behaviour of flows for which $|q| \ll |k|$. For plane Poiseuille flow, $|q/k| = 0{\cdot}17$. In HS2 a theory was presented for point-centred bursts in three dimensions. It is not possible to adjust this theory directly to the more complicated equations now seen to be relevant to this situation. For when $q = 0$, $|A|^2$ takes the form of a function of $L\xi^2+M\eta^2$ and τ near the burst, where L and M are known functions of τ. When $q \neq 0$, we must also determine B and it follows from (2.31) that B and $|A|^2$ cannot both be of this form. A possible structure for point-centred bursts has been found by Hocking (1974), who also gives the results of some numerical calculations, limited to the case when all the coefficients in (2.34) are real.

There is one class of disturbances for which the general equation can be reduced to the simpler two-dimensional form. Suppose that A can be written as a function of χ and τ, where

$$\chi = l\xi + m\eta, \quad l^2+m^2 = 1, \tag{3.1}$$

and l and m are constants, corresponding to a disturbance whose amplitude, at a fixed value of τ, is constant along lines inclined to the flow direction. We refer to this type of disturbance as a skewed two-dimensional disturbance. It follows immediately from (2.31) that

$$B = m^2|A|^2, \tag{3.2}$$

and hence (2.34) reduces to

$$\frac{\partial A}{\partial \tau} - (a_2 l^2 + b_2 m^2)\frac{\partial^2 A}{\partial \chi^2} = \frac{d_1}{d_{1r}}A + (k+m^2 q)|A|^2 A. \tag{3.3}$$

The properties of this equation have already been extensively studied in HS1. In identifying the various types of behaviour of the solution in that paper, the equation was written with scaled variables in the form

$$\frac{\partial A}{\partial \tau} - (1+ia_i)\frac{\partial^2 A}{\partial \xi^2} = \frac{d_1}{d_{1r}}A + (1+i\bar{\delta}_i)|A|^2 A. \tag{3.4}$$

Thus the diagram showing the regions in the $\bar{\delta}_i, a_i$ plane appropriate to different types of behaviour (HS1, figure 1) can be used in connexion with the complex conjugate of (3.3) if we put

$$a_1 = -\frac{a_{2i}l^2 + b_{2i}m^2}{a_{2r}l^2 + b_{2r}m^2}, \quad \bar{\delta}_i = -\frac{k_i + m^2 q_i}{k_r + m^2 q_r}. \tag{3.5}$$

The position of the Poiseuille line in that diagram must now be changed. It identifies the behaviour of two-dimensional skewed disturbances in plane Poiseuille flow, but the correction to the equation and the adjusted value of a_{2i} change its position to a hyperbolic curve between the points (5·63, −0·147) and (4·87, −17·3). The transition point between solutions which burst and those which remain finite for all τ is now given by $m = 0{\cdot}842$, so that all two-dimensional disturbances skewed at angles greater than 57·3° or 1·00 rad to the direction of the undisturbed flow end by bursting according to the present theory. The previous value was 56°.

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